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Periodic motions of a reversible second-order mechanical system Application to the Sitnikov problem☆

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Abstract

A theory of the symmetric periodic motions (SPMs) of a reversible second-order system is presented which covers both oscillations and rotations. The structural stability property of the generating autonomous reversible system, which lies in the fact that the presence or absence of SPMs in a perturbed system is independent of the actual form of the "reversible" perturbations, is established. Both the case of the generation of SPMs from the family of SPMs of the generating system and birth cycle from the equilibrium state are investigated. Criteria of Lyapunov stability in a non-degenerate situation are obtained for the SPMs which are generated (in case of small values of the parameter). A method is proposed for constructing and investigating the Lyapunov stability of all the SPMs. The conditions for the existence of a cycle (symmetric and asymmetric) in the neighbourhood of a support "almost" resonance SPM are established for all cases of resonances. The theoretical results are applied to a study of the motion of a particle along a straight line which passes through the centre of mass of the system perpendicular to the plane of the identical attracting and simultaneously radiating main bodies (an extension of the Sitnikov problem) in the photogravitational version of the three-body problem. The circular problem is analysed and two different series of families of SPMs are found in the weakly elliptic problem. The instability of the equilibrium state is proved in the case of parametric resonance and the stability (and instability) domains are distinguished for arbitrary values of the eccentricity. All the SPMs with a period of 2π are constructed and the property of Lyapunov stability is investigated for these motions.

The problem of a pendulum with a vibrating suspension point,[1] the Sitnikov problem[2] and the Beletskii problem,[3] where the "naturalness and simplicity of the formulations of the problems are combined with the extraordinary richness and diversity of their content",[4] primarily belong to the outstanding simplest model problems associated with the analysis of the structure of the phase space of a dynamic problem. Numerous papers (see the reviews Refs. 5–8, for example) have been devoted to these problems, the flow of which has not been exhausted, while both the initial problems as well as their modifications are investigated. Of recent investigations, we mention the papers on pendulum problems,[9–16] the Sitnikov problem[17–22] and on the plane motions of a satellite about a centre of mass under the action of factors of a different nature[7,23–29] (see also, the bibliography in the above-mentioned papers).

The model problems are described by a periodic second-order equation, the distinctive property of which is its invariance under the replacement (of the phase coordinates and time) of (x, x', t) by $(x, -x', -t)$ or $(-x, x', -t)$. Here, in the problem of a pendulum with a vertically (horizontally) vibrating suspension point and in the Sitnikov problem,

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the equation is invariant under the two transformations simultaneously. It is well known that, in dynamics, reversibility manifests itself as an invariance under a transformation $\mathbf{G}$, $\mathbf{G}^2 = \mathrm{id}$ (id is the identity transformation), of the phase space with a simultaneous reversal of the sign of the time.[30] In mechanical systems, $\mathbf{G}$ is a linear transformation, and these systems form a class of reversible mechanical systems.[31–33]

The so-called symmetric periodic motions (SPMs) of the type of oscillations and rotations are systematically investigated below for a reversible, periodic second-order system, and problems of the existence, construction and stability of SPMs and the problem of the birth of a cycle near a specified SPM are solved. Results regarding SPMs are given in the photogravitational version of the Sitnikov problem.

1. Symmetric periodic motions and their stability

Consider the fairly smooth second-order reversible system, 2π-periodic in t

$$\dot{u} = U(u, v, t), \quad \dot{v} = V(u, v, t) \tag{1.1}$$

with the fixed set $\mathbf{M} = \{u, v, t : v = 0, \sin\ t = 0\}$.[34] In the case of the functions U and V, which are 2π-periodic in v, the fixed set will be $\mathbf{M}^* = \{u, v, t : \sin\ v = 0, \sin\ t = 0\}$.[34]

The functions U and V can also satisfy the conditions

$$U(-u, v, -t) = U(u, v, t), \quad V(-u, v, -t) = -V(u, v, t).$$

System (1.1) then has two fixed sets.

We will call the solution $u = \varphi(t)$, $v = \psi(t)$ of system (1.1) a $2\pi k$-periodic motion, $k \in \mathbb{N}$, if

$$\varphi(t + 2\pi k) = \varphi(t), \quad \psi(t + 2\pi k) = \psi(t) + 2\pi m, \quad m \in \mathbb{Z}. \tag{1.2}$$

Note that, in mechanical systems, the solution (1.2) describes both the oscillatory motion ($m = 0$) and the rotational motions ($m \neq 0$).[34] Here, a rotation can be a forward rotation ($m > 0$) or a reverse rotation ($m < 0$). The motion (1.2) in the problem of the rotation of a satellite in an elliptic orbit (the Beletskii problem[3]) has been called[a] a generalized periodic solution (also, see Ref. 7).

System (1.1) can contain a parameter μ. Suppose that, when $\mu = 0$, system (1.1) becomes an autonomous system. Then, when $\mu \neq 0$, we obtain the problem of the periodic motions of the reversible system

$$\dot{u} = U_0(u, v) + \mu U_1(\mu, u, v, t), \quad \dot{v} = V_0(u, v) + \mu V_1(\mu, u, v, t) \tag{1.3}$$

which is close to an autonomous system.

When $\mu = 0$, we have a generating system, and the functions μU_1 and μV_1 are called perturbations.

In a typical case, the periodic motions of an autonomous system form a family.[35] This is the rule[36] for motions of a reversible system which are symmetric with respect to the set $\mathbf{M}$ ($\mathbf{M}^*$ in the case of a system which is 2π-periodic with respect to v). The corresponding necessary and sufficient conditions for existence of the motions (1.2) in an autonomous reversible system are well known.[34,37]

The formulation of the problem of the continuation of periodic motions with respect to a parameter and the method of solving it belong to Poincaré;[38] the method of continuation with respect to a parameter, proposed for the first time for problems in celestial mechanics,[38] was subsequently developed in detail for analytical systems of general form.[35] Two cases arise when solving a problem:[39] (a) the structurally stable case, when a property of the system of having periodic motions is solely determined by the generating system (it is called structurally stable in the sense of periodic motions) and is independent of the actual form of the perturbations, (b) the non-structurally stable case when, to solve the problem, an examination of the perturbation is necessary.

Undoubtedly, the solution of a problem depends both on the class to which the generating system belongs as well as the class of the perturbations; these classes are determined by the content of the actual problem.

In a problem concerning the continuation of a $2\pi k$-periodic motion, selected from the family of periodic motions of an autonomous system, we have the non-structurally stable case as a rule.

[a] Varin VP. Generalized periodic solutions of the equations of the oscillations of a satellite. Preprint No. 97. Moscow: Inst Prikl. Mat. Ross. Akad. Nauk; 1997.

In the simplest Hamiltonian system

$$\ddot{z} = \mu(1 + z^2)(1 + \cos^2 t) \tag{1.4}$$

the autonomous generating equation $\ddot{z} = 0$ admits of a family of periodic motions $z = c$ (const). However, when $\mu \neq 0$, not any solution of Eq. (1.4) is periodic (for a periodic solution, the right-hand side of Eq. (1.4) must vanish at a certain instant of time t). Example (1.4) leads to an important conclusion: an autonomous Hamiltonian system is not structurally stable in the sense of periodic motions when the perturbations preserving the Hamiltonian character of the system are considered.

Eq. (1.4) is also reversible. Here, the family of periodic motions $z = c$, $\dot{z} = 0$ belongs to the fixed set $\mathbf{M} = \{z, \dot{z}, t : \dot{z} = 0, \sin\ t = 0\}$. However, not any periodic motion from $\mathbf{M}$ is continued with respect to the parameter μ. Hence, reversibility is also not a guarantee of the structural stability of the generating system in the sense of periodic motions in the class of perturbations which preserve the property of reversibility.

On the other hand, the periodic motions of a reversible system which is close to a conservative system with one degree of freedom have been studied[34] and the structural stability of the generating conservative system in the sense of the periodic motions in the class of reversible perturbations has been established. The problem of the structural stability of a reversible autonomous generating system in the general situation was also considered[40] and the necessary conditions for structural stability in the sense of periodic motions were obtained for symmetric motions. For a second-order system, these conditions turn out to be also sufficient conditions in the general situation.

Note that there is no analogue of the above mentioned assertion in a Hamiltonian system (see example (1.4)) if the system is not simultaneously reversible with a corresponding fixed set.

The following interesting problem involves an investigation of the stability of the symmetric periodic motions (SPMs) which are found for small μ. According to Poincaré's theorem,[38] when $\mu = 0$ one characteristic exponent is equal to zero. In the case of reversible system (1.3), we obtain two such exponents.[41] At the same time, in the case of a family of SPMs of a generating system, as a rule we have a Jordan cell.[40] For small $\mu \neq 0$, the characteristic exponents $\pm\kappa$ are close to zero. Real values of $\pm\kappa$ lead to instability by the first approximation.[42] Lower-order resonances are precluded in the case of pure imaginary $\pm\kappa$ of small modulus, and Lyapunov stability therefore holds.[43]

Below, to determine the numbers $\pm\kappa$ for small $\mu \neq 0$, we use a specially constructed approximate solution for a linear system and the results obtained earlier in Ref. 44.

If the parameter is not explicitly separated out in system (1.1), all the SPMs are found by the method described earlier in Ref. 34. The stability of the SPMs is then investigated on the basis of the variational equations

$$\delta\dot{u} = \frac{\partial U}{\partial u}\delta u + \frac{\partial U}{\partial \upsilon}\delta\upsilon, \quad \delta\dot{\upsilon} = \frac{\partial V}{\partial u}\delta u + \frac{\partial V}{\partial \upsilon}\delta\upsilon; \quad U = U(\varphi, \psi, t), \quad V = V(\varphi, \psi, t). \tag{1.5}$$

In the non-degenerate case, Lyapunov stability follows from the existence of pure imaginary roots if there are no resonances of up to the fourth order inclusive.[43,45] Resonance cases are investigated on the basis of well-known results.[46,47]

For the numerical determination of the characteristic exponents of the second-order system (1.5), it is sufficient to construct just a single solution of Cauchy's problem with an initial point $\delta u_1(0) = 1$, $\delta\upsilon_1(0) = 0$ or $\delta u_2(0) = 0$, $\delta\upsilon_2(0) = 1$ in the period $2\pi k$.[44] Then,

$$\pm\kappa = \frac{1}{2\pi k}\operatorname{arcch}\delta u_1(2\pi k) = \frac{1}{2\pi k}\operatorname{arcch}\delta\upsilon_2(2\pi k). \tag{1.6}$$

Hence, the construction and investigation of the stability property of all the SPMs of reversible system (1.1) consists of finding all the starting points for the SPMs by the method described previously in Ref. 34, solving Cauchy's problem in a period of time t with the now known initial conditions for the combined system consisting of Eqs. (1.1) and (1.5), and calculating the exponents $\pm\kappa$ by formula (1.6).

Note that problems concerning the plane motions of a satellite in an elliptic orbit about a centre of mass have been investigated using this scheme in Refs. 23–28.

In a situation which is close to a resonance one, cycles occur in the neighbourhood of the SPM. This problem has been studied recently in the case of an autonomous system of general form.[48] The conditions for the birth of a cycle in the neighbourhood of the SPM of a reversible system are obtained below.

The investigation of SPM in the Sitnikov problem[2] is worthy of special attention. This problem has been studied in detail in the plan of logically possible situations.[20] A photogravitational formulation of the Sitnikov problem is considered in Section 8 in which all the 2π- periodic SPMs are constructed and their stability is investigated. The results for the Sitnikov problem follow from this as a special case.

2. A quasi-autonomous system

We precede the proof of the main result for a quasi-autonomous system with a lemma.

Lemma. *If a smooth reversible system*

$$\dot{u} = U_0(u, v), \quad \dot{v} = V_0(u, v); \quad U_0(u, -v) = -U_0(u, v), \quad V_0(u, -v) = V_0(u, v) \tag{2.1}$$

allows of a SPM $(u(t), v(t))$ *with period T, which does not coincide with the equilibrium position, and* $v(0) = 0$, *then*

$$\dot{v}(T/2) \neq 0.$$

Proof. The phase portrait of the reversible system (2.1) is symmetrical about the *Ou* axis and each trajectory intersecting this axis is described by an even function $u(t)$ and an odd function $v(t)$ (Fig. 1; *a* is an oscillation and *b* is a rotation). The variational equations are reversible[46] in the neighbourhood of such a trajectory and have the solution: $p(t) = \dot{u}(t)$, $q(t) = \dot{v}(t)$. In the case of a *T*-periodic motion, the functions $p(t)$, $q(t)$ are *T*-periodic, where $p(t)$ is an odd function and $q(t)$ is an even function. Hence,

$$p(T/2) = \dot{u}(T/2) = 0, \quad q(T/2) = \dot{v}(T/2) \neq 0. \quad \square$$

Theorem 1. *If the generating autonomous system obtained from system (1.3) when* $\mu = 0$, *possesses a single-parameter (with respect to the parameter h) symmetric family of periodic motions of period* $T(h)$, $T(h^*) = 2\pi$ *such that*

$$dT(h^*) \neq 0, \tag{2.2}$$

then, for small $\mu \neq 0$, *system (1.3) has a unique symmetric* 2π*-periodic motion which is generated from the* 2π*-periodic motion of the generating system. The fact of the existence of this motion depends solely on the properties of the generating system and is independent of the form of the perturbations* μU_1, μV_1.

Proof. Suppose $u(\mu, u^0, v^0, t)$, $v(\mu, u^0, v^0, t)$ is the solution of system (1.3) with starting point (u^0, v^0) when $t = 0$. The necessary and sufficient condition for the existence of a symmetric 2π-periodic motion has the form.[34]

$$v(\mu, u^0, 0, \pi) = v(0, u^0, 0, \pi) + \mu v_1(\mu, u^0, 0, \pi) = \pi m, \quad m \in \mathbb{Z} \tag{2.3}$$

$(u(0, u^0, 0, t), v(0, u^0, 0, t)$ is the symmetric solution of the generating system). $\square$

Suppose $\mu = 0$. Eq. (2.3) admits of the solution $u^0 = u^*$ (const).

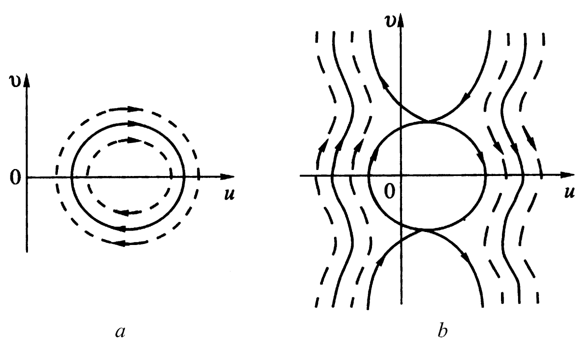

Fig. 1.

The necessary and sufficient conditions for the existence of a symmetric 2τ-periodic motion in the case of a generating autonomous system are written as follows:

$$v(0, u^0, 0, \tau) = \pi m, \quad m \in \mathbb{Z}. \tag{2.4}$$

Here, by assumption, Eq. (2.4) has a family of solutions $u^0 = \chi(h)$, $\tau = T(h)/2$, that is,

$$v(0, \chi(h), 0, T(h)/2) = \pi m, \quad m \in \mathbb{Z}$$

($\chi(h^*) = u^*$, $T(h^*) = 2\pi$). Differentiating this relation at the point $h = h^*$ and using condition (2.2) and the inequality $\partial v(0, u^*, 0, \pi)/\partial t \neq 0$, in accordance with the lemma we conclude that $\partial v(0, u^*, 0, \pi)/\partial u^0 \neq 0$. We now apply the implicit function theorem to Eq. (2.3): an interval exists containing a zero and the unique continuous function $\lambda(\mu)$, defined in this interval, is such that

$$v(\mu, \lambda(\mu), 0, \pi) = \pi m, \quad m \in \mathbb{Z}; \quad \lambda(0) = u^*.$$

Consequently, for small $|\mu| \neq 0$, a solution of system (1.3), for which $u^0 = \lambda(\mu)$, exists and has a period 2π.

We now point out an important special case. Suppose a system is close to a conservative system with one degree of freedom. We consider the equation

$$\ddot{z} + f(z) = \mu F(\mu, z, \dot{z}, t), \quad F(\mu, z, \dot{z}, t + 2\pi) = F(\mu, z, \dot{z}, t). \tag{2.5}$$

Here, Theorem 1 is applicable in two cases

1) $f(-z) = f(z), \quad F(\mu, -z, \dot{z}, -t) = -F(\mu, z, \dot{z}, t)$

2) $F(\mu, z, -\dot{z}, -t) = F(\mu, z, \dot{z}, t)$.

In the case of Eq. (2.5), the result of Theorem 1 has been established earlier in the first case[34] and in the second case.[40] For rotations in system (2.4), a result has been obtained in the first case[34] which also holds when condition (2.2) is violated.

3. Construction of the symmetric periodic motions (SPMs)

Suppose the generating autonomous system, obtained from system (1.3) when $\mu = 0$, allows of a family of SPMs which depends on the parameter h,

$$u = \varphi(h, t), \quad v = \psi(h, t); \quad \varphi(h, t) = \varphi(h, -t), \quad \psi(h, t) = -\psi(h, t). \tag{3.1}$$

For $h = h^*$, the period of the solution $T(h^*)$ is equal to 2π. In the non-degenerate case (condition (2.2) is satisfied when $\mu \neq 0$), we represent the 2π-periodic SPM (Theorem 1) as follows:

$$u = \varphi + \mu\varphi_1 + \mu^2\varphi_2 + o(\mu^2), \quad v = \psi + \mu\psi_1 + \mu^2\psi_2 + o(\mu^2) \tag{3.2}$$

(φ_j are even functions and ψ_j are odd functions of t). Then, substituting solution (3.2) into system (1.3) we obtain

$$\begin{aligned} &\dot{\varphi}_j = A_0^-(t)\varphi_j + A_0^+(t)\psi_j + f_j(t), \quad \dot{\psi}_j = B_0^+(t)\varphi_j + B_0^-(t)\psi_j + g_j(t); \quad j = 0, 1, 2 \\ &A_0^- = U_u(\varphi, \psi), \quad A_0^+ = U_v(\varphi, \psi), \quad B_0^+ = V_u(\varphi, \psi), \quad B_0^- = V_v(\varphi, \psi) \\ &f_0 = g_0 = 0, \quad f_1 = U_1(0, \varphi, \psi, t), \quad g_1 = V_1(0, \varphi, \psi, t) \\ &f_2 = U_{1\mu 0} + U_{1u0}\varphi_1 + U_{1v0}\psi_1, \quad g_2 = V_{1\mu 0} + V_{1u0}\varphi_1 + V_{1v0}\psi_1. \end{aligned} \tag{3.3}$$

The subscripts u, v and μ denote a partial derivative with respect to the corresponding variable and a zero subscript denotes substitution of the quantities 0, φ and ψ into the partial derivatives instead of μ, u and v.

According to Theorem 1, each of the systems (3.3) has a unique SPM with period 2π.

System (3.3) has a fundamental solution matrix

$$\begin{Vmatrix} \varphi_h(h, t) & \dot{\varphi}(h, t) \\ \psi_h(h, t) & \dot{\psi}(h, t) \end{Vmatrix} \quad \left(\chi_h = \frac{\partial \chi_h}{\partial h}\right)$$

with a determinant

$$\Delta = \Delta(0)\exp\left(\int_0^t (A^- + B^-)dt\right)$$

It can be verified that the substitution

$$x = -\frac{1}{2\pi\Delta}T'(\varphi_j\dot{\psi} - \psi_j\dot{\varphi}), \quad y = \frac{1}{\Delta}[\varphi_j\eta(t) - \psi_j\xi(t)]; \quad j = 0, 1, 2 \quad \left(T' = \frac{dT(h^*)}{dh}\right) \tag{3.4}$$

(the corresponding even function ($\xi(t)$) and odd function ($\eta(t)$) have a period 2π) reduces system (3.3) when $j = 0$ to the system

$$\dot{x} = 0, \quad \dot{y} = x.$$

In fact, we have

$$\varphi_0 = C_1^0\dot{\varphi} + C_2^0\varphi_h, \quad \psi_0 = C_1^0\dot{\psi} + C_2^0\psi_h$$

(C_1^0 and C_2^0 are constants), whence we obtain

$$C_1^0 = \frac{1}{\Delta}(\varphi_0\psi_h - \psi_0\varphi_h), \quad C_2^0 = -\frac{1}{\Delta}(\varphi_0\dot{\psi} - \psi_0\dot{\varphi}). \tag{3.5}$$

Next, taking into account the fact that the functions

$$\varphi\left(h, \frac{T(h)t}{2\pi}\right), \quad \psi\left(h, \frac{T(h)t}{2\pi}\right)$$

have a period equal to 2π, independent of h, we calculate the derivatives

$$\xi(t) = \frac{1}{2\pi}T't\dot{\varphi}(h^*, t) + \varphi_h(h^*, t), \quad \eta(t) = \frac{1}{2\pi}T't\dot{\psi}(h^*, t) + \psi_h(h^*, t). \tag{3.6}$$

from them. Then, from relations (3.5) and (3.6), we obtain

$$C_1^0 = -\frac{1}{2\pi}T'tC_2^0 + \frac{1}{\Delta}(\varphi_0\eta - \psi_0\xi). \tag{3.7}$$

Finally, for x, we choose $(T'/(2\pi))C_2^0$. We then obtain the equation for y by differentiating the equality (3.7).

As a result of the substitution (3.4), system (3.3) takes the form

$$\dot{x} = -\frac{T'}{2\pi\Delta}(f_j\dot{\varphi} - g_j\dot{\psi}), \quad \dot{y} = x + \frac{1}{\Delta}[f_j\eta(t) - g_j\xi(t)]. \tag{3.8}$$

It is easy to derive the conditions for the existence of a SPM from this

$$x(0) + \int_0^\pi \frac{1}{\Delta}[f_j\eta(t) - g_j\xi(t)]dt = 0 \tag{3.9}$$

and to find the initial point $x(0)$ for this SPM. The solution is then constructed in an explicit form by taking the quadrature.

Note that the explicit formulae obtained from relation (3.7) enable one to solve the problem of synthesizing the SPM constructively.

4. The stability of symmetric periodic motions (SPMs)

We will now set up the system of variational equations for the SPMs

$$\dot{p} = A^-(t,\mu)p + A^+(t,\mu)q, \quad \dot{q} = B^+(t,\mu)p + B^-(t,\mu)q$$
$$A^{\pm}(t,\mu) = A_0^{\pm}(t) + \mu A_1^{\pm}(t) + \mu^2 A_2^{\pm}(t) + o(\mu^2) \tag{4.1}$$
$$B^{\pm}(t,\mu) = B_0^{\pm}(t) + \mu B_1^{\pm}(t) + \mu^2 B_2^{\pm}(t) + o(\mu^2).$$

Here,

$$A_1^- = U_{uu*}\varphi_1 + U_{uv*}\psi_1 + U_{1u*}, \quad A_1^+ = U_{vu*}\varphi_1 + U_{vv*}\psi_1 + U_{1v*}$$
$$A_2^- = U_{1uu*}\varphi_2 + U_{1uv*}\psi_2 + (U_{uuu*}\varphi_1^2 + 2U_{uuv*}\varphi_1\psi_1 + U_{uvv*}\psi_1^2)/2$$
$$A_2^+ = U_{1vu*}\varphi_2 + U_{1vv*}\psi_2 + (U_{vuu*}\varphi_1^2 + 2U_{vvu*}\varphi_1\psi_1 + U_{vvv*}\psi_1^2)/2.$$

The expressions for B_k^+ (B_k^-) are obtained from the expressions for A_k^- (A_k^+) by replacing U by V. The plus (minus) superscript denotes even (odd) functions and an asterisk indicates that

$$\mu = 0, \quad u = \varphi(h^*, t), \quad v = \psi(h^*, t).$$

has been put in the calculated partial derivatives.

We will seek the solution of system (4.1) in the form

$$p = p_0(t) + \mu p_1(t) + \mu^2 p_2(t) + o(\mu^2), \quad q = q_0(t) + \mu q_1(t) + \mu^2 q_2(t) + o(\mu^2). \tag{4.2}$$

We then have the following system of equations for the functions $p_k(t)$, $q_k(t)$ $(k=0, 1, 2)$

$$\dot{p}_k = A_0^-(t)p_k + A_0^+(t)q_k + F_k(t), \quad \dot{q}_k = B_0^+(t)p_k + B_0^-(t)q_k + Q_k(t), \tag{4.3}$$

where

$$F_0 = Q_0 = 0, \quad F_1 = A_1^-(t)p_0 + A_1^+(t)q_0, \quad Q_1 = B_1^+(t)p_0 + B_1^-(t)q_0$$
$$F_2 = A_1^-(t)p_1 + A_1^+(t)q_1 + A_2^-(t)p_0 + A_2^+(t)q_0$$
$$Q_2 = B_1^+(t)p_1 + B_1^-(t)q_1 + B_2^+(t)p_0 + B_2^-(t)q_0.$$

Systems (4.3) enable us to find successively a solution of the form (4.2) which satisfies the conditions

$$p(0) = 0, \quad q(0) = 1. \tag{4.4}$$

To do this, using formulae similar to (3.4)

$$x_k = -\frac{T'}{2\pi\Delta}(p_k\dot{\psi} - q_k\dot{\varphi}), \quad y_k = \frac{1}{\Delta}(p_k\eta - q_k\xi) \tag{4.5}$$

we change from systems (4.3) to the following systems

$$\dot{x}_k = -\frac{T'}{2\pi\Delta}(F_k\dot{\psi} - Q_k\dot{\varphi}), \quad \dot{y}_k = x_k + \frac{1}{\Delta}(F_k\eta - Q_k\xi). \tag{4.6}$$

The transformation inverse to (4.5) then has the form

$$p_k = -\frac{2\pi}{T'}x_k\xi - y_k\dot{\varphi}, \quad q_k = -\frac{2\pi}{T'}x_k\eta - y_k\dot{\psi}. \tag{4.7}$$

In formulae (4.3), (4.5)–(4.7), the subscript k takes the values 0, 1 and 2.

We will now show that it follows from formulae (4.5)–(4.7) that zero values of x_k and y_k correspond to zero values of p_k and q_k, and $x_0(0)=0$, $y_0(0)=1$ correspond to the values (4.4). We also note that Eq. (4.6) can be integrated in explicit form in quadratures.

We will now write out the solutions of systems (4.6)

$$x_0 = 0, \quad y_0 = 1$$

$$\begin{aligned} x_1 &= -\frac{T'}{2\pi}\int\frac{1}{\Delta}(A_1^+\dot\psi - B_1^-\dot\varphi)dt, \quad y_1 = \int\left[x_1 + \frac{1}{\Delta}(A_1^+\eta - B_1^-\xi)\right]dt \\ x_2 &= -\frac{T'}{2\pi}\int\frac{1}{\Delta}(F_2^*\dot\psi - Q_2^*\dot\varphi)dt, \quad y_2 = \int\left[x_2 + \frac{1}{\Delta}(F_2^*\eta - Q_2^*\xi)\right]dt \end{aligned} \tag{4.8}$$

$$F_2^* = A_1^-p_1 + A_1^+q_1 + A_2^+, \quad Q_2^* = B_1^+p_1 + B_1^-q_1 + B_2^-.$$

In the expression for x_1, the integrand is an even periodic function and $\dot\varphi(0) = 0$. Hence, representing the functions by their Fourier series, we evaluate an indefinite integral. We have

$$x_1(t) = -\frac{T'}{2\pi}[a_0t + \chi^-(t)], \quad a_0 = \int_0^{2\pi}\frac{1}{\Delta}(A_1^+\dot\psi - B_1^-\dot\varphi)dt \tag{4.9}$$

($\chi^-(t)$ is an odd periodic function). Then, for y_1, taking account of the oddness of the integrand here, we obtain

$$y_1(t) = -\frac{T'}{2\pi}\left[\frac{a_0t^2}{2} + \theta^+(t)\right], \quad \theta^+(0) = 0 \tag{4.10}$$

($\theta^+(t)$ is an even periodic function).

Formulae (4.7), (4.9) and (4.10) enable us to calculate

$$q_1(2\pi) = a_0^* = T'\pi a_0\dot\psi(h^*, 0). \tag{4.11}$$

Then, in the case when $a_0 = 0$, formulae (4.7), (4.9) and (4.10) give

$$p_1(t) = \chi^-(t)\xi + \frac{T'}{2\pi}\theta^+(t)\dot\varphi, \quad q_1 = \chi^-(t)\eta + \frac{T'}{2\pi}\theta^+(t)\dot\psi. \tag{4.12}$$

Substituting expressions (4.12) into the relation (4.8) for x_2 and y_2 and reasoning in the same way as for x_1 and y_1, we obtain

$$q_2(2\pi) = a_1^* = T'\pi a_1\dot\psi(h^*, 0), \quad a_1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\Delta}(F_2^*\dot\psi - Q_2^*\dot\varphi)dt. \tag{4.13}$$

Hence, the calculations give

$$q(2\pi) = 1 + \mu q_1(2\pi) + \mu^2q_2(2\pi) + o(\mu^2)$$

(the numbers $q_1(2\pi)$ and $q_2(2\pi)$ are defined by formulae (4.11) and (4.13)). It follows from this[44] that, when $\mu a_0^* > 0$, the characteristic exponents $\pm\kappa$ will be pure imaginary and, when $\mu a_0^* < 0$, they will be real and of opposite sign. In the case when $a_0 = 0$, the sign of the number a_1^* gives pure imaginary ($a_1^* < 0$) or real ($a_1^* > 0$) exponents κ.

For small $|\mu| > 0$, the characteristic exponents are also close to zero. Lower-order resonances are not realized in the system in the case of pure imaginary numbers. This means that the Lyapunov stability of the SPM, the existence of which was established in Theorem 1, follows from the pure imaginary exponents in the non-degenerate case. Non-degeneracy implies that the real coefficient C in the normal form, written in the complex-conjugate variables z and $\bar z$,

$$\dot z = \kappa z + iCz^2\bar z + \dots$$

is equal to zero.

Theorem 2. *In the non-degenerate case* ($C \neq 0$, $a_0^* \neq 0$), *satisfaction of the inequality* $\mu a_0^* < 0$ *is a necessary and sufficient condition for the Lyapunov stability of the SPM* (3.2). *When* $a_0^* = 0$, $a_1^* \neq 0$, $C \neq 0$, *the inequality* $a_1^* < 0$ *will be such a condition*.

5. A special case

Suppose $U_0 = 0$ and $V_0 = u$ in system (1.3). Then, the system being generated admits of a unique zero equilibrium state belonging to the fixed set $\{u, v : v = 0\}$. For small $\mu \neq 0$, we have the problem of an SPM in the neighbourhood of an equilibrium.

Theorem 1 is inapplicable to the system

$$\dot{u} = \mu U_1(\mu, u, v, t), \quad \dot{v} = u + \mu V_1(\mu, u, v, t) \tag{5.1}$$

as the generating system does not allow of the required family. We will write the necessary and sufficient conditions for the existence of a 2π-periodic, symmetric solution in the form[37]

$$v(\mu, u^0, 0, \pi) = 0. \tag{5.2}$$

When $\mu = 0$, equality (5.2) is obtained by integration of the linear system and has the form $u^0\pi = 0$. Hence, when $\mu \neq 0$, Eq. (5.2) always has a unique solution $u^0 = u^0(\mu)$. This conclusion is independent of the form of the specific perturbations U_1 and V_1.

Hence, in the neighbourhood of the point $u = v = 0$, the reversible system (5.1) always has a unique SPM[39]

$$u = \mu u_1(t) + o(\mu), \quad v = \mu v_1(t) + o(\mu)$$

$$\begin{aligned} u_1(t) &= u^* + \int_0^t U_1(0, 0, 0, t)dt, \quad u^* = \text{const} \\ v_1(t) &= \int_0^t \left[u^* + \int_0^\tau U_1(0, 0, 0, \nu)d\nu + V_1(0, 0, 0, \tau) \right] d\tau, \end{aligned} \tag{5.3}$$

regardless of the form of the specific perturbations U_1 and V_1, and the constant u^* is determined from the condition for the function $v_1(t) : v_1(\pi) = 0$ to be periodic.

In the system of variational Eq. (4.1), we have

$$A_0^-(t) \equiv 0, \quad A_0^+(t) \equiv 0, \quad B_0^+(t) \equiv 1, \quad B_0^-(t) \equiv 0.$$

Hence, system (4.3) is integrated without a preliminary change to the variables x_k and y_k. We obtain

$$p_1 = \int A_1^+(t)dt, \quad q_1 = \int [p_1(t) + B_1^-(t)]dt$$

$$p_2 = \int [A_1^-(t)p_1 + A_1^+(t)q_1 + A_2^+(t)]dt, \quad q_2 = \int [p_2 + B_1^+ p_1 + B_1^- q_1 + B_2^-]dt.$$

Now, representing the functions $A_1^+(t)$ and $B_1^-(t)$ by their Fourier series, we will calculate

$$p_1 = a_0 t + \chi^-(t), \quad q_1 = \frac{a_0 t^2}{2} + \theta^+(t); \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} A_1^+(t)dt, \quad \theta^+(0) = 0 \tag{5.4}$$

($\chi^-(t)$ and $\theta^+(t)$ are odd and even periodic functions respectively).

In the case when $a_0 = 0$, the formulae for p_2 and q_2 are analogous to (5.4). Only, here, the mean value is

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} [A_1^-(t)\chi^-(t) + A_1^+(t)\theta^+(t) + A_2^+(t)]dt.$$

Theorem 3. *The reversible system (5.1) always has a SPM and, moreover, it is unique. The SPM is stable when $\mu a_0 < 0$ and unstable when $\mu a_0 > 0$. When $a_0 = 0$, the stability property of the SPM is determined by the sign of the number a_1: it is stable when $a_1 < 0$ and unstable when $a_1 > 0$.*

Example. We will consider a special case of a well-known problem, that is, a pendulum with a vibrating suspension point when there is no gravitational force

$$\ddot{z} = -\mu\cos t\sin z.$$

It is obvious that, for any μ, the pendulum has two equilibrium positions: $z=0$ and $z=\pi$.

According to Theorem 3, these periodic motions are also observed for small μ.

We will investigate the stability of the equilibrium. From formulae (4.1), we have

$$A_1^-(t) = 0, \quad A_1^+(t) = -\cos t$$

$$B_1^+(t) = B_1^-(t) = A_2^-(t) = A_2^+(t) = B_2^+(t) = B_2^-(t) \equiv 0.$$

Hence,

$$p_1 = t\sin t + \cos t - 1, \quad q_1 = -t(1+\cos t) + 2\sin t; \quad a_0 = 0, \quad a_0' = -\pi < 0,$$

and the equilibrium is stable.

When $\mu=0$, the pendulum allows of a uniform 2π-periodic rotation $z = nt$, $n \in \mathbb{N}$. We shall assume that $z=nt+\theta$. Then, for θ, we obtain the equation

$$\ddot{\theta} = -\mu\cos t\sin(nt+\theta).$$

According to Theorem 3, in the case of small $\mu \neq 0$, 2π-periodic oscillations, which are specified by the formulae (5.3):

$$\theta = \begin{cases} \dfrac{\mu}{2}\left[\dfrac{\sin(n+1)t}{(n+1)^2} + \dfrac{\sin(n-1)t}{(n-1)^2}\right] + o(\mu), & n > 1 \\ -\dfrac{\mu}{4}\sin 2t + o(\mu), & n = 1. \end{cases} \tag{5.5}$$

are superimposed on the uniform rotations.

We will investigate the stability of the motions (5.5). Of all the functions (4.1),

$$A_1^+(t) = \begin{cases} -\cos^2 t, & n = 1 \\ -\dfrac{1}{2}[\cos(n-1)t + \cos(n+1)t], & n > 1 \end{cases}$$

is the only function which is not identically equal to zero.

When $n=1$, we calculate that $a_0 = -\pi < 0$. This means that, when $\mu \geq 0$ ($\mu < 0$), a rotation close to a uniform rotation with $n=1$ is stable (unstable).

When $n>1$, we have $a_0=0$. We introduce the notation

$$s_{\alpha\beta}^{\pm}(t) = \frac{\sin^\alpha(n\pm1)t}{2(n\pm1)^\beta}, \quad c_{\alpha\beta}^{\pm}(t) = \frac{\cos^\alpha(n\pm1)t}{2(n\pm1)^\beta}$$

and calculate

$$p_1 = -(s_{11}^- + s_{11}^+)t - (c_{12}^- + c_{12}^+) + b$$

$$q_1 = (c_{12}^- + c_{12}^+)t + (s_{13}^- + s_{13}^+) + bt$$

$$b = \frac{(n^2+1)}{(n-1)^2(n+1)^2}, \quad a_1 = -\frac{1}{8\pi}\int_0^{2\pi}\left[\frac{\cos^2(n-1)t}{(n-1)^2} + \frac{\cos^2(n+1)t}{(n+1)^2}\right]dt < 0.$$

The stability of the motions (5.3) when $n>1$ follows from this; it is obvious that the property of stability is independent of the sign of μ.

6. The birth of a cycle in the neighbourhood of the SPM of an "almost" resonance system

It is also possible to give the following interpretation to the results in Sections 2–4. The one parameter family of systems (1.3) (μ is the parameter) is considered. When $\mu = 0$, system (1.3) allows of a family of SPMs in whose neighbourhood cycles occur in the general situation when $\mu \neq 0$. The stability of these cycles depends on the sign of μ.

The result in Section 5 is also treated in a similar manner. Only, here, we have an equilibrium state which belongs to a fixed set instead of a family of SPMs.

A cycle can also occur in a system in the neighbourhood of a "support" SPM, if the latter is "almost" resonance motion. Here, it should be borne in mind that system (1.1) depends on the parameter ε and allows of a family of SPMs with respect to the parameter ε and, when $\varepsilon = 0$, the SPM is a resonance motion. Then, when $\varepsilon \neq 0$, we have an isolated, "almost" resonance SPM. In the neighbourhood of the "support" (null) SPM, we have an "almost" resonance system.

A closely related problem has been considered earlier for an autonomous system.[48,49]

When considering a cycle in an "almost" resonance system, we will use the general assertion concerning the existence of periodic motions in a reversible system with a small parameter (Ref. 49, Theorem 1) and also the normal form. Note that the resulting normal form enables one to carry out a "complete" classification of the phase portraits for a system in a "general position" which requires a separate treatment. The classification is well known in the case of Hamiltonian systems.[50]

In the neighbourhood of a chosen SPM, the system also has the form of (1.1) with the sole refinement that $U(0, 0, t) = V(0, 0, t) = 0$. This means that the system has a null SPM.

We shall assume that the characteristic exponents $\pm\lambda$ are pure imaginary and that the system is an "almost" resonance system.

$$\lambda = \lambda_0 + i\varepsilon, \quad P\lambda_0 = iq, \quad P \in \mathbb{N}, \quad q \in \mathbb{Z}; \quad \lambda_0 = \text{const.} \tag{6.1}$$

The normal form then depends on ε and, also, on the form of the resonance. We will use a normalizing transformation which is continuous with respect to ε.[51]

After these preliminary remarks, we will now analyse particular cases of resonance.

1°. $P > 4$. In this case, the normal form in the first non-linear approximation is independent of the order of the resonance. In the complex-conjugate variables η and $\bar{\eta}$, we have

$$\dot{\eta} = \lambda\eta + iC_{11}(\varepsilon)\eta^2\bar{\eta} + H(\varepsilon, \eta, \bar{\eta}, t)$$

(C_{11} is a real number). We now make the substitution

$$\eta = w\exp(\lambda_0 t), \quad \bar{\eta} = \bar{w}\exp(-\lambda_0 t).$$

Then,

$$\dot{w} = i\varepsilon w + iC_{11}(\varepsilon)w^2\bar{w} + W(\varepsilon, w, \bar{w}, t).$$

Finally, we write

$$\dot{r} = R(\varepsilon, t, \theta, t), \quad \dot{\theta} = \varepsilon + C_{11}(\varepsilon)r^2 + H(\varepsilon, r, \varphi, t)$$

in radius - angle variables.

We now change the scale and apply a well-known result (Ref. 49, Theorem 1). The cycle is then determined from the amplitude equation

$$\varepsilon + C_{11}(0)r^{0^2} = 0, \quad r^0 = \text{const}$$

and represents a SPM. This SPM is isolated and only exists if $\varepsilon C_{11}(0) > 0$.

Hence, when $P > 4$, a cycle always exists and moreover, it is unique. The period of this SPM is equal to $2\pi P$.

2°. $P = 4$ (fourth-order resonance). We will write the normal form in the complex-conjugate variables w and $\bar{w}$ (Ref. 46)

$$\dot{w} = i\varepsilon w + i[C_{11}(\varepsilon)w^2\bar{w} + C_{-1,3}(\varepsilon)\bar{w}^3] + W(\varepsilon, w, \bar{w}, t)$$

(C_{11} and $C_{-1,3}$ are constants). The corresponding amplitude equation then has the form

$$\varepsilon + (C_{11}^0 + C_{-1,3}^0\cos 4\theta^0)r^0 = 0, \quad \sin 4\theta^0 = 0 \tag{6.2}$$

(Henceforth, a zero superscript indicates that the value of a coefficient is taken when $\varepsilon = 0$). When $|C_{11}^0| \neq |C_{-1,3}^0|$, the system always allows of cycles. The cycles represent isolated SPMs with period 8π.

It is clear from Eq. (6.2) that, in the neighbourhood of a null support SPM ($|C_{11}^0| > |C_{-1,3}^0|$)[46] when $\varepsilon \neq 0$, 8 cycles occur, 4 cycles occur in the opposite case when $\varepsilon C_{11}(0) > 0$, and the remaining four cycles when $\varepsilon C_{11}^0 < 0$.

3°. $P = 3$ (third-order resonance). In the case being considered, the normal form has a simple form[46]

$$\dot{w} = i\varepsilon w + iC(\varepsilon)\bar{w}^2 + W(\varepsilon, w, \bar{w}, t),$$

and the amplitude equation is

$$\varepsilon + C^0\cos 3\theta^0\sqrt{r_1^0} = 0, \quad \sin 3\theta^0 = 0.$$

The support SPM is unstable[46] and 6 cycles with period 6π (3 cycles when $\varepsilon > 0$ and 3 cycles when $\varepsilon < 0$) occur in the neighbourhood of the support SPM.

4°. $P = 2$, $|q|$ is an odd number (parametric resonance). In this case, the support SPM, as a rule, is unstable by the first approximation. The normal form in the variables w and $\bar{w}$ has the form

$$\dot{w} = i\varepsilon\bar{w} + i[C_{20}(\varepsilon)w^2 + C_{02}(\varepsilon)\bar{w}^2 + C_{11}(\varepsilon)w\bar{w} + C_{-1,3}(\varepsilon)w^{-1}\bar{w}^3]w + W(\varepsilon, w, \bar{w}, t)$$

(C_{jk} are real constants). We now change to radius - angle variables

$$\begin{aligned} \dot{r} &= 2\varepsilon r\sin 2\theta + 2(C_{-}\sin 2\theta - C_{-1,3}\sin 4\theta)r^2 + R(\varepsilon, r, \theta, t) \\ \dot{\theta} &= \varepsilon\cos 2\theta + (C_{11} + C_{+}\cos 2\theta + C_{-1,3}\cos 4\theta)r + H(\varepsilon, r, \theta, t) \\ C_{\pm} &= C_{02} \pm C_{20}. \end{aligned} \tag{6.3}$$

The existence of SPM in the form of a cycle is then revealed using the amplitude equation

$$\varepsilon\cos 2\theta^0 + (C_{11}^0 + C_{+}^0\cos 2\theta^0 + C_{-1,3}\cos 4\theta^0)r^0 = 0 \tag{6.4}$$

($\sin 2\theta^0 = 0$). From this, we obtain

$$r^0 = -\frac{\varepsilon}{C_{11}^0 \pm C_{+}^0 + C_{-1,3}}.$$

The plus sign corresponds to a cycle $\theta^0 = 0$, π and the minus sign to a cycle $\theta^0 = \pi/2$, $3\pi/4$.

It is seen that two cycles occur for each sign of ε and that these cycles of SPM have a period 4π.

A special feature of the resonance being considered is the existence of an asymmetric cycle. We now set up the system of amplitude equations[48] for system (6.3). Eq. (6.4) will be one of these equations. The other equation is

$$\varepsilon + (C_{-} - 2C_{-1,3}\cos 2\theta^0)r^0 = 0. \tag{6.5}$$

We express ε in terms of this equation and substitute it into Eq. (6.4). We obtain the quadratic equation

$$C_{11}^0 - C_{-1,3}^0 + (C_+^0 - C_-^0)\cos 2\theta^0 + 4C_{-1,3}\cos^2 2\theta^0 = 0, \qquad (6.6)$$

from which we determine one or two values of $\cos 2\theta^0$ which satisfy the condition $|\cos 2\theta^0| < 1$.

Hence, when the obvious conditions which are imposed on the coefficients of Eqs. (6.5) and (6.6) are satisfied, the system has 4 or 8 asymmetric cycles with period 4π.

5°. $P = 1$ (principal resonance). The normal form in the case of this resonance is distinguished from the case when $P = 2$ by the non-linear terms and has the form

$$\dot{w} = i\varepsilon\bar{w} + i(C_{10}w^2 + C_{01}w\bar{w} + C_{-1,2}\bar{w}^2) + W(\varepsilon, w, \bar{w}, t).$$

In radius - angle variables, we then obtain

$$\dot{r} = 2\varepsilon r\sin 2\theta + 2(C_-\sin 2\theta - C_{-1,2}\sin 3\theta)r\sqrt{r} + R(\varepsilon, w, \bar{w}, t)$$

$$\dot{\theta} = \varepsilon\cos 2\theta + (C_+\cos\theta + C_{-1,2}\cos 3\theta)\sqrt{r} + H(\varepsilon, w, \bar{w}, t)$$

$$C_\pm = C_{01} \pm C_{10}.$$

We now set up the system of amplitude equations

$$\begin{aligned} &\varepsilon\sin 2\theta^0 + (C_-^0\sin\theta^0 + C_{-1,2}^0\sin 3\theta^0)\sqrt{r^0} = 0 \\ &\varepsilon\cos 2\theta^0 + (C_+^0\cos\theta^0 + C_{-1,2}^0\cos 3\theta^0)\sqrt{r^0} = 0. \end{aligned} \qquad (6.7)$$

The solutions of system (6.7), when $\sin\theta^0 = 0$, determine the symmetric cycles

$$r^0 = \left(\frac{\mp\varepsilon}{C_+^0 + C_{-1,2}}\right)^2, \quad \cos\theta^0 = \pm 1.$$

It is obvious that, when $C_+^0 \neq C_{-1,2}^0$, the cycle is unique. The cycle represents SPM with period 2π.

The simple roots of system (6.7), for which $\sin\theta^0 \neq 0$, determine the asymmetric cycles

$$r^0 = \left[\frac{-2\varepsilon\cos\theta^0}{C_-^0 + C_{-1,2}^0(1 + 2\cos 2\theta^0)}\right]^2, \quad \cos 2\theta^0 = \frac{C_-^0 - C_+^0}{C_+^0 - C_{-1,2}^0}. \qquad (6.8)$$

There are four such cycles and the period of the motion along them is equal to 2π.

We now summarize the conclusions of Section 6.

Theorem 4. *Cycles representing SPMs with period $2P\pi$ always occur in the almost resonance system (6.1). Asymmetric cycles also occur together with a symmetric cycle in the case of parametric resonance ($P = 2$) and the principal resonance ($P = 1$).*

7. Possible scenarios for the birth of cycles

1°. The equilibrium of an autonomous system is stable in the linear approximation and the system is an "almost" resonance system. When $\mu \neq 0$ (periodic perturbations), symmetric cycles occur and asymmetric cycles also occur in the case when $P = 1, 2$ (Theorem 4).

2°. The same holds for the neighbourhood of the SPMs of a periodic system (Theorem 4).

3°. An autonomous system has a unique equilibrium which belongs to a fixed set. When $\mu \neq 0$, a unique cycle occurs (Theorem 3).

4°. A generating autonomous system has a family of SPMs for which the period $T(h)$ depends on the parameter h. When $\mu \neq 0$, a unique cycle occurs if the perturbations are 2π-periodic, $dT(2\pi) \neq 0$.

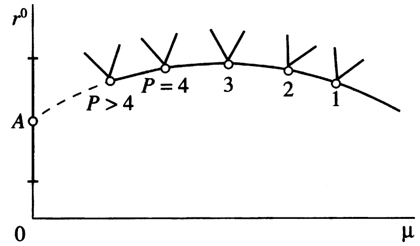

Fig. 2.

We now present a diagram which reflects the birth of cycles in a system with a parameter μ. When $\mu = 0$, the system is autonomous and has a family of SPMs with period $T(h)$ (which is picked out in Fig. 2 by the segment on the r^0 axis). The condition $dT(2\pi) \neq 0$ is satisfied at point A. For small $\mu > 0$, cycles occur which depend on the parameter μ. For small μ, the characteristic exponents $\pm\kappa$ are close to zero. On the branch corresponding to the stable reference SPM (after the segment of the curve shown by a dashed line), the open circles represent resonance situations, and cycles (with period $2P\pi$) occur here. The successively spaced resonance points represent the monotonic dependence of $\pm\kappa$ on μ.

8. A special version of the photogravitational three-body problem (the Sitnikov problem)

We will now consider the motion of a particle **P** along a fixed straight line OZ in a gravitationally repulsive field of two identical attracting and simultaneously radiating bodies $\mathbf{S}_1$ and $\mathbf{S}_2$ (a double star). The main bodies $\mathbf{S}_1$ and $\mathbf{S}_2$ rotate with respect to one another in elliptic orbits in the OXY plane and the OZ line passes through their centre of mass O perpendicular to the OXY plane (Fig. 3).

The motion of the particle **P** is described by the periodic second-order equation[22]

$$z'' + \frac{z}{1 + e\cos\upsilon}\left(e\cos\upsilon + \frac{Q}{R^3}\right) = 0, \quad R^2 = \frac{1}{4} + z^2 \tag{8.1}$$

(e and υ are the eccentricity and the true anomaly in the problem of two bodies $\mathbf{S}_1$ and $\mathbf{S}_2$, Q is the reduction coefficient, which characterizes the radiating action of the double star on the particle ($Q \leq 1$), z is the distance of the particle from the centre of mass and a prime denotes a derivative with respect to υ). Eq. (8.1) is reversible in the sense that it is invariant with respect to the substitutions $(z, z', \upsilon) \to (\pm z, \mp z', -\upsilon)$, and it has two fixed sets.

Note that the equation of motion of the three-body photogravitational problem[52,53] allows of an integral manifold in which $x = y = 0$ and the z coordinate varies in accordance with Eq. (8.1).[22]

When $Q = 1$, the main bodies do not radiate. In this problem, Sitnikov[2] proved the existence of oscillatory motions, and Alekseyev found the "possibility of using the methods of symbolic dynamics"[4] and solved the Chazy problem of the final motions in the four-body problem.[4,54] On the other hand, Eq. (8.1) is an example of the simplest non-integrable system which is rich in content in all respects. Hence, numerous papers (for a brief review, see Ref. 8) are devoted

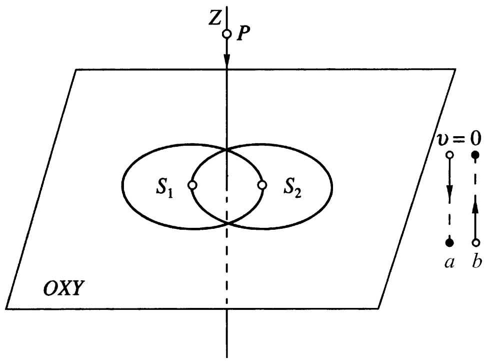

Fig. 3.

to an investigation of Eq. (8.1) and they continue to appear.[17–22]. Problem (8.1) when $Q = 1$ is called the Sitnikov problem.[8,17–21]

The purpose of the discussion below is to investigate all possible SPMs in problem (8.1) with an arbitrary physically admissible parameter Q ($Q \leq 1$). The theory developed in Sections 2–6 is systematically applied here.

1°. *The circular problem* ($e = 0$). In this case, we have a conservative system with one degree of freedom and an energy integral

$$z'^2 = 2\left(h + \frac{Q}{R}\right), \quad h = \text{const.}$$

It means that, when $Q > 0$ (the Newton attraction force exceeds the light pressure), the particle executes oscillations along the OZ line, which are symmetrical about the point O if $h < 0$, and, when $h \geq 0$, all the motions of the particle are departing motions as when $t \to +\infty$.

In the case when $Q < 0$, we have $h > 0$. We obtain three types of motions:

1) the particle arrives from the point $z = +\infty(-\infty)$, approaches a finite distance to the point O and departs to $+\infty(-\infty)$,
2) the particle moves from $+\infty(-\infty)$ to $-\infty(+\infty)$,
3) the particle enters point O when $t \to +\infty(-\infty)$.

We will now describe an ensemble of particles $\mathbf{P}_j$ which are arranged on the OZ axis. Since the reduction coefficient Q in the three-body photogravitational problem is not solely dependent on the radiating properties of the bodies $\mathbf{S}_1$ and $\mathbf{S}_2$ but, also, on the characteristics of an individual particle $\mathbf{P}_j$, then, in the ensemble being considered, a particle $\mathbf{P}_j$ moves in accordance with Eq. (8.1) and each with its own reduction coefficient Q_j. The energy of a particle h_j is determined in a random manner by the initial conditions. As a result, an ensemble of chaotically moving particles is "observed". Such an effect is generated by particles, several of which oscillate, each of them having their own frequency, some others move in the OXY plane and others move away from this plane, each of them doing so non-uniformly at an individual rate.

2°. *The weakly elliptic problem* ($0 < e \ll 1$). Here, we will use the conclusions of Theorems 1 and 2 in the analysis of the symmetric periodic orbits of the particle.

First, we make an important observation. The reversible system (8.1) has two fixed sets. It can therefore allow of motions which are symmetrical with respect to the set $\mathbf{M}_1 = \{z, \dot{z}, \upsilon : \dot{z} = 0, \sin\ \upsilon = 0\}$ or to the set $\mathbf{M}_2 = \{\dot{z}, \upsilon : z = 0, \sin\ \upsilon = 0\}$. Some of these motions can be simultaneously symmetrical with respect to both sets, that is, they are doubly symmetrical motions (see also Ref 20). Note that the family of oscillations in the circular problem consists of doubly symmetrical motions.

We calculate the period of the oscillatory motions when $e = 0$

$$T = 4\int_0^{z_0} \frac{dz}{\sqrt{2(h + Q/R)}}, \quad z_0^2 = \frac{h^2}{Q^2} - \frac{1}{4}, \quad h < 0, \quad Q > 0$$

(z_0 is the amplitude of the oscillations). We put

$$2z = \text{tg}\theta, \quad 2z_0 = \text{tg}\theta_0 \quad (0 < \theta, \theta_0 < \pi/2).$$

Then,

$$T = \frac{1}{\sqrt{Q}}\int_0^{\theta_0} \frac{d\theta}{\cos^2\theta\sqrt{\cos\theta - \cos\theta_0}}.$$

We now make the replacement

$$\sin\frac{\theta}{2} = ku, \quad \sin\frac{\theta_0}{2} = k, \quad \cos\theta - \cos\theta_0 = 2k(1-u^2), \quad d\theta = \frac{2kdu}{\sqrt{1-k^2u^2}}.$$

As a result, we have the following final expression

$$T = \sqrt{\frac{2}{Q}}\int_0^1 \frac{du}{f(k,u)}; \quad f(k,u) = (1-2k^2u^2)^2\sqrt{(1-u^2)(1-k^2u^2)}, \tag{8.2}$$

which gives the explicit relation $T = T(Q, k)$.

The period T is a function of the amplitude of the oscillations (in the parameter k). It is clear from expression (8.2) that $dT/dk > 0$ always. Moreover, $T(Q, k) \to \infty$ when $k \to \sqrt{2}/2$. In accordance with Theorem 1 when $e > 0$, two SPMs with period $2\pi q$ arise in the weakly elliptic case for each $q \in \mathbb{N}$. One of them is symmetrical with respect to the set $\mathbf{M}_1$ and the other is symmetrical with respect to the set $\mathbf{M}_2$.

Theorem 5. *In the weakly elliptic Sitnikov problem (8.1) with $Q > 0$, two SPMs with period $2\pi q$ arise for each $q \in \mathbf{N}$. When $v = 0$, in the first motion we have $z(0) = z_0 \neq 0$, $z'(0) = 0$ (Fig. 3, case a) and, in the second motion, $z(0) = 0$, $z'(0) = z'_0 \neq 0$ (Fig. 3, case b).*

3°. Stability of the equilibrium position. Eq. (8.1) allows of an obvious, null equilibrium. In the initial three-body photogravitational problem,[52,53] this solution corresponds to an internal collinear libration point.

Eq. (8.1) reduces to the simple form

$$\ddot{\zeta} = -\frac{fmQ\zeta}{(\zeta^2 + r^2)^{3/2}}, \quad \zeta = rz \tag{8.3}$$

(r is the distance between the bodies $\mathbf{S}_1$ and $\mathbf{S}_2$ in the elliptic motion and a dot denotes differentiation with respect to time t, m is the total mass of the bodies and f is the gravitational constant). The instability of the equilibrium position when $Q < 0$ follows from this.

When $Q > 0$, the equilibrium is stable in the weakly elliptic problem if there is no parametric resonance. The frequency of the oscillations in the circular problem is equal to $\omega = 2\sqrt{2Q}$, and this means that, when $Q = 1/32$, the frequency becomes the resonance frequency ($2\omega = 1$).

In the neighbourhood of the equilibrium, we obtain a Mathieu equation of the form

$$z'' + [\omega^2 - (\omega^2 - 1)e\cos v]z = 0.$$

Parametric resonance therefore leads to instability.

We will now investigate the stability of the equilibrium position in the elliptic problem. To do this, we construct a solution of Cauchy's problem in the interval $v \in [0, 2\pi]$ and, using formula (1.6), calculate the characteristic exponents. We then apply the theorem on stability in non-degenerate situations.[43]. As a result, the stability (instability) domains and the resonance curves are distinguished in the (e, Q) plane (Fig. 4).

4°. *SPMs in the elliptic problem*. Theorem 5 guarantees the existence of two SPMs of period $2\pi q (q \in \mathbf{N})$ for small e. The condition $2\sqrt{2Q}q \geq 1$, which determines the lower bound of Q for the occurrence of SPMs, follows from expression (8.2). So, there are only SPMs of period 2π in the problem when $Q \geq 1/8$.

The periodic motions found in the weakly elliptic problem can be continued numerically for finite values of the eccentricity e. However, it is preferable here to use the method in Ref. 34 which enables one to construct all the SPMs of the problem. When account is taken of the existence of two fixed sets in Eq. (8.1), use of the above method leads to the construction of two families of SPMs from the parameter e for each fixed $q \in \mathbf{N}$.

Note that the method in Ref. 34 not only makes it possible to continue the SPMs which have been determined in Theorem 5, but also to find all the other SPMs.

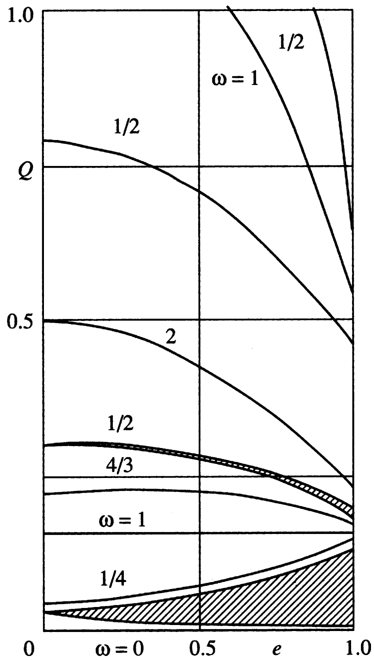

Fig. 4.

The necessary and sufficient conditions for the existence of the first family of SPMs have the form

$$z(0) = z_0 \text{ (const)}, \quad z'(0) = 0, \quad z'(\pi q) = 0. \tag{8.4}$$

For the second family, we obtain

$$z(0) = 0, \quad z'(0) = z_0' \text{ (const)}, \quad z(\pi q) = 0. \tag{8.5}$$

When $z_0 = z_0' = 0$, we have a trivial SPM (an equilibrium).

We now consider the process of constructing a motion which satisfies conditions (8.4). We subdivide the interval $[0, a]$ of the z axis by the points z_{0k} $(k=0, 1, \ldots, l)$: $0 = z_{00} < z_{01} < \ldots < z_{0l} = a$ and, from each point z_{0k} when $v = 0$, we let out a trajectory with a velocity $z_k'(0) = 0$. Then, when $v = \pi q$, the ends of these trajectories belong to the curve Γ, and the points $\Gamma_k = \{z_k(\pi k), z_k'(\pi k)\} \subset \Gamma$, $\Gamma_0 = \{0, 0\}$.

If A is the point of intersection of the curve Γ with the z axis, we have $z_k'(\pi k) z_j'(\pi k) < 0$ at the points Γ_k and Γ_j adjacent to it. This condition establishes the fact that an SPM exists. The accuracy with which the SPM can be constructed is determined by the method chosen to solve Cauchy's problem. The choice of the points z_{0k} is made using the golden section-method. The scheme for investigating the stability of an SPM has been described in Section 1.

The results of the investigation of SPMs with period 2π are presented for $Q = 1$ (the Sitnikov problem). The initial points z_0 (see Fig. 3, case a) for the motions (8.4), which form a family Σ with respect to the parameter e are given in Fig. 5. This family consists of three subfamilies Σ_1, Σ_2, Σ_3. The subfamily Σ_1 arises when $e = 0$ from the 2π-periodic SPM of the circular problem when $e > 0$ and is stable as long as $e < e_* < 0.55$. There is subsequently a change in the stability and, when $e > e_*$, the SPMs of the subfamily Σ_1 become unstable. When $e = 0$, a subfamily Σ_2 arises from the SPMs with period $2\pi/3$, and, when $e > 0$, it consists of unstable SPMs. A change in the stability of the SPMs in Σ_2 also occurs at the point $e = e_*$. The third subfamily, Σ_3, is created from the local SPMs when $e = e_*$, and it is unstable.

The law of a change in stability for a fixed value of e holds in the curves of the family Σ. When $e < e_*$, the sign of the stability in the curves Σ_1 and Σ_2 is different (the family Σ_1 is stable and Σ_2 is unstable), when $e = e_*$ the subfamily Σ_3 arises, and, when $e > e_*$, we have that the family Σ_1 is unstable and the family Σ_2 is stable.

The question of the double symmetry of the SPMs constructed is of interest. It has been discussed in Ref. 20 from the point of view of the existence of logically possible SPMs. The phase curves for SPMs in the half-plane $z' \geq 0$ are presented in Fig. 6 for characteristic values of e. The subfamilies Σ_1 and Σ_2 consist of SPMs which are symmetrical solely about the z axis (of the fixed set $\mathbf{M}_1$) while the SPMs in the curve Σ_3 are symmetrical both about the z axis and

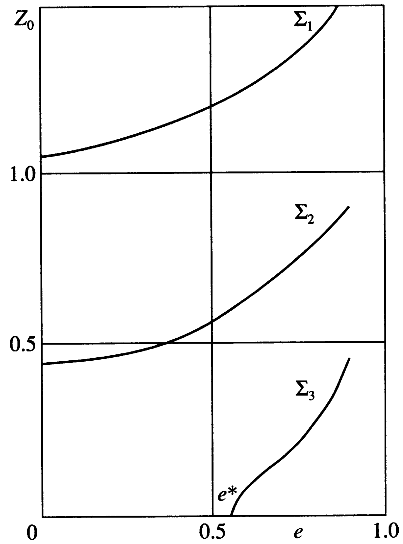

Fig. 5.

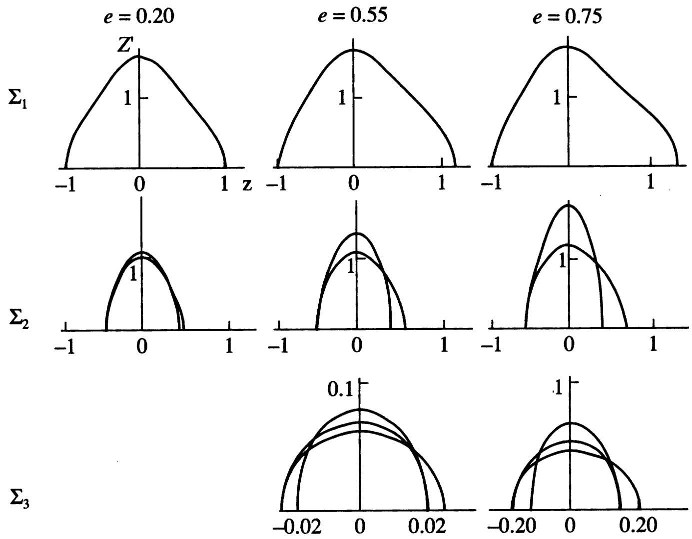

Fig. 6.

the z' axis (of the sets $\mathbf{M}_1$ and $\mathbf{M}_2$). This is not surprising as Theorem 1 guarantees the uniqueness of the continuation of the subfamilies Σ_1 and Σ_2 which do not have branch points. As far as the subfamily Σ_3 is concerned, it arises when $e = e_* \neq 0$ from a double symmetrical SPMs and remains as such.

It is interesting to compare the results for the family Σ with the results for the family Σ^* which satisfies conditions (8.5). These results were also obtained but omitted here due to lack of space.

The problem of investigating all the SPMs with period $2\pi q$ $(q > 1)$ is interesting relation to possible bifurcations and also in relation to unstable SPMs, which have initial points close to the initial points for oscillatory motions.[2]

Finally, a study of all the SPMs in the photogravitational version of the Sitnikov problem $(0 < Q < 1)$ is of interest.

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